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Analytic Study of Chaos of the Tent Map: Band Structures, Power Spectra, and Critical Behaviors

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Chaotic behaviors of the tent map (a piecewise-linear, continuous map with a unique maximum) are studied analytically throughout its chaotic region in terms of the invariant density and the power spectrum. As the height of the maximum is lowered, successive band-splitting transitions occur in the chaotic region and accumulate to the transition point into the nonchaotic region. The time-correlation function of nonperiodic orbits and their power spectrum are calculated exactly at the band-splitting points and in the vicinity of these points. The method of eigenvalue problems of the Frobenius–Perron operator is used. 2^{m-1} critical modes, where $m = 1, 2, 3, \ldots$, are found which exhibit the critical slowing-down near the 2^{m-1}-band to 2^m-band transition point. After the transition these modes become periodic modes which represent the cycling of nonperiodic orbits among 2^m bands together with the periodic modes generated by the preceding band splittings. Scaling laws near the transition point into the nonchaotic region are investigated and a new scaling law is found for the total intensity of the periodic part of the spectrum.

KEY WORDS: Chaos; mapping; invariant measure; ergodicity; band structure of chaos; power spectrum of chaos; critical behavior; scaling law; Frobenius–Perron operator.

1. INTRODUCTION

In a previous paper,[(1)] we have studied the single-band regime and the band-splitting transition to the two-band regime in the chaotic region of a tent map. Especially, the critical behavior near this transition point has been investigated in detail. The present paper deals with the overall

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structure of the chaotic region of the tent map. The successive band-splitting transitions in the chaotic region are clearly described in terms of the invariant density, and chaotic structures of the 2^m-band regimes ($m = 0, 1, 2, \ldots$) are studied in terms of the power spectrum. Critical behaviors are investigated near the 2^{m-1}-band to 2^m-band transition point and near the transition point to the nonchaotic region. All of these are made analytically on the basis of rigorous calculations of appropriate quantities.

A map which has a single quadratic maximum, for example, the logistic model, exhibits a transition to chaos via a cascade of period-doubling bifurcations.(2,3) By taking the logistic model, Feigenbaum has predicted certain universal properties for the onset of chaos.(4,5) The period-doubling sequence as a precursor to chaos has now been found in many systems which are described by low-dimensional nonlinear ordinary differential equations.(6–11) Some of them have definite physical ground and the other are simple mathematical models. The onset properties are in good agreement with Feigenbaum's predictions.(6,7,9,11)

The period-doubling route to chaos has also been observed in the transition to turbulent convection in Bénard cells with small aspect ratios.(12–14) The experimental evidence is in favor of Feigenbaum's predictions, though more detailed studies seem to be needed.

All these studies and others indicate that certain essential aspects of chaotic behaviors can be understood in terms of simple one-dimensional maps.(15–17) This is the reason why we take up simple one-dimensional maps in this paper.

At present, however, relatively little is known about chaotic properties all over the chaotic region. Quadratic maps have chaotic bands which split successively as the control parameter is varied toward the chaotic transition point.(3) The band-splitting transitions have also been found in low-dimensional dynamical systems.(10,18) In the vicinity of the chaotic transition point, some scaling laws have been found for the Lyapunov exponent(19,20) and the power spectrum.(21,22) The purpose of this paper is to study, all over the chaotic region, the properties of the chaos which has the band structure. We do this by taking the tent map as a typical and soluble model. The scaling laws are also discussed on the basis of a rigorous analysis for the tent map.

The tent map is a map f_a on the unit interval $J = [0, 1]$ into itself:

$$f_a(x) = \begin{cases} ax, & x \in [0, 1/2] \\ a(1-x), & x \in (1/2, 1] \end{cases} \tag{1.1}$$

where a is a control parameter which is varied between 0 and 2. We consider a discrete process generated by

$$x_n = f_a(x_{n-1}) = f_a^{(n)}(x_0) \qquad (n = 1, 2, 3, \ldots) \tag{1.2}$$

where $f_a^{(n)}$ denotes the nth iterate of f_a. The Lyapunov exponent is given by

$$\lambda_a(x) = \lim_{n\to\infty} \frac{1}{n} \ln\left| \frac{d}{dx} f_a^{(n)}(x) \right| = \ln a \tag{1.3}$$

everywhere in J. If $0 < a < 1$, then for every point $x \in J$ the orbit $f_a^{(n)}(x)$ converges to the unique fixed point 0 as n increases. At $a = 1$, every point $x \in [0, 1/2]$ is a fixed point. The chaotic region is $1 < a \leqslant 2$, in which $\lambda_a > 0$.

For a map to exhibit chaos, it is necessary that the map has the stretching and folding properties.(17) The simplest of such maps which have the band structure is the tent map. It should be noted, however, that the transition to chaos in this map occurs abruptly at $a = 1$ since the period-doubling bifurcations degenerate into one point $a = 1$.

The power spectrum studied in this paper is the Fourier–Laplace transform of the time-correlation function of nonperiodic orbits. In calculating the correlation function, we adopt the method of eigenvalue problems of the Frobenius–Perron operator presented by Mori, So, and Ose,(23) although we use the Frobenius–Perron operator itself instead of the modified one proposed by them. The method is developed systematically in Appendix B.

In Section 2, the band structure is discussed and the invariant density is given in the 2^m-band regime ($m = 0, 1, 2, \ldots$). The correlation function and the power spectrum at the band-splitting point $\bar{a}_m$ are obtained in Section 3, and those for the two sequences of values of the parameter a which converge to $\bar{a}_m$ from above and from below are presented in Section 4. In Section 5, the critical behaviors near the band-splitting point $\bar{a}_m$ are discussed. Those near $\bar{a}_1$ have been studied in the previous paper.(1) The critical behaviors near the chaotic transition point $a = 1$ are investigated in Section 6. The last section is devoted to a summary and some remarks on the scaling laws in the vicinity of the chaotic transition point.

2. BAND SPLITTINGS AND THE INVARIANT DENSITY

In the chaotic region, the tent map f_a has a unique ergodic invariant measure which is absolutely-continuous with respect to the Lebesgue measure.(3,24) The density function of the invariant measure (the invariant density) has been obtained by Ito, Tanaka, and Nakada.(25) In this section, we discuss the band structure in the chaotic region, and give the formula for the invariant density in the 2^m-band regime in terms of the density in the single-band regime.

By virtue of ergodicity, the invariant density $\rho_a(x)$ for f_a in the interval J is determined as a unique solution to the equation[3]

$$\mathscr{H}\rho_a(x) = \rho_a(x) \tag{2.1}$$

where $\mathscr{H}$ is the Frobenius–Perron operator defined by

$$\mathscr{H}F(x) \equiv \int_J dy\, F(y)\delta(f_a(y) - x) \tag{2.2}$$

Some properties of $\mathscr{H}$ are given in Appendix A.

In the chaotic region $1 < a \leqslant 2$, the map f_a has a unique fixed point $x^*(a)$ other than 0:

$$x^*(a) = a/(a+1) \tag{2.3}$$

This fixed point is always unstable. We write

$$x_n(a) \equiv f_a^{(n)}(1/2) \tag{2.4}$$

Then $x_1(a) = a/2$ and $x_2(a) = a(1 - a/2)$. The intervals $(0, x_2(a))$ and $(x_1(a), 1)$ are transient for f_a, and thus $\rho_a(x) = 0$ in these intervals. We have $f_a A = A$ for $A = [x_2(a), x_1(a)]$. If $\sqrt{2} < a \leqslant 2$, then A is an attractor and $\rho_a(x)$ is positive for every $x \in A$.[25] Here $\sqrt{2}$ is the value of a satisfying $x_3(a) = x^*(a)$. At $a = \sqrt{2}$, the attractor A splits into two bands at the position $x = x^*(a)$. This is seen as follows.

Let I_0 and I_1 be the intervals $I_0 = [x^*(a), y_0(a)]$ and $I_1 = [y_1(a), x^*(a)]$, where $y_0(a)$ and $y_1(a)$ are the values of x which satisfy $f_a^{(2)}(x) = x^*(a)$. See Fig. 1. Define the two transformations $\varphi_{ia} : I_i \to J$ $(i = 0, 1)$, where

$$\begin{aligned}
&\varphi_{ia}(x) = \alpha_i(a)[x - x^*(a)] \\
&\alpha_0(a) = [y_0(a) - x^*(a)]^{-1} = a(a+1)/(a-1) \\
&\alpha_1(a) = [y_1(a) - x^*(a)]^{-1} = -(a+1)/(a-1)
\end{aligned} \tag{2.5}$$

Then, for $1 < a \leqslant \sqrt{2}$, the two maps $f_a^{(2)} | I_i : I_i \to I_i$ and $f_{a^2} : J \to J$ are conjugate:

$$f_{a^2} = \varphi_{ia} \circ f_a^{(2)} \circ \varphi_{ia}^{-1} \qquad (i = 0, 1) \tag{2.6}$$

Therefore, in terms of ρ_{a^2} for f_{a^2} in J, the invariant density $\rho_a^{(2)}(x; I_i)$ for $f_a^{(2)} | I_i$ in I_i is given as follows:

$$\rho_a^{(2)}(x; I_i) = |\alpha_i(a)|\rho_{a^2}(\varphi_{ia}(x)) \tag{2.7}$$

Indeed, we see that (2.7) satisfies the equation

$$\int_{I_i} dy\, \rho_a^{(2)}(y; I_i)\delta\big(f_a^{(2)}(y) - x\big) = \rho_a^{(2)}(x; I_i) \tag{2.8}$$

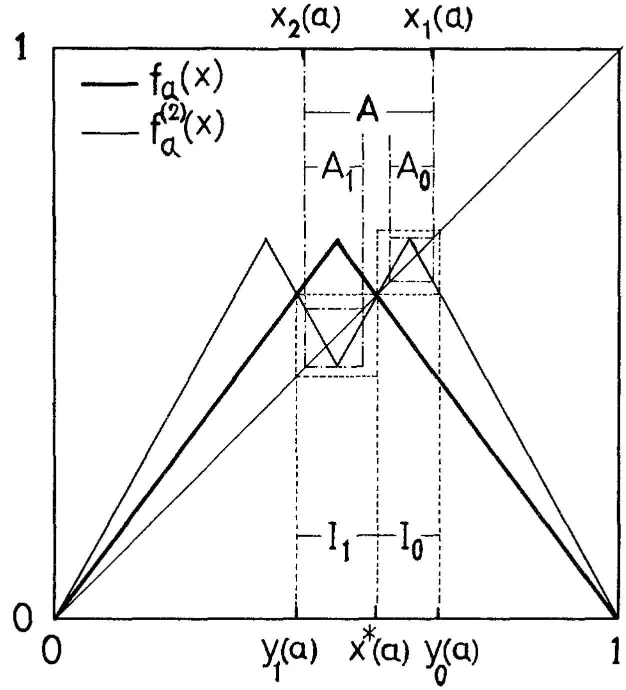

Fig. 1. Tent map $f_a(x)$ (thick line) and the iterated map $f_a^{(2)}(x)$ (thin line) in the 2-band regime. The intervals A_0 and A_1 are the two bands.

Using (2.2) and (2.8), we also have $\mathscr{H}\rho_a^{(2)}(x; I_0) = \rho_a^{(2)}(x; I_1)$ and $\mathscr{H}\rho_a^{(2)}(x; I_1) = \rho_a^{(2)}(x; I_0)$. Hence we obtain for $1 < a \leqslant \sqrt{2}$

$$\rho_a(x) = \sum_{i=0,1} \frac{|\alpha_i(a)|}{2} \rho_{a^2}(\varphi_{ia}(x)) \tag{2.9}$$

If $1 < a < \sqrt{2}$, the intervals corresponding to $0 < \varphi_{ia}(x) < x_2(a^2)$, namely, $(x^*(a), x_3(a))$ and $(x_4(a), x^*(a))$ are transient for f_a, and therefore $\rho_a(x) = 0$ for $x_4(a) < x < x_3(a)$. Let A_i $(i = 0, 1)$ be the intervals corresponding to $x_2(a^2) \leqslant \varphi_{ia}(x) \leqslant x_1(a^2)$. Then we have $f_a A_0 = A_1$ and $f_a A_1 = A_0$ for $1 < a \leqslant \sqrt{2}$. At $a = \sqrt{2}$, the attractor A for $\sqrt{2} < a \leqslant 2$ splits into two bands A_0 and A_1. See Fig. 1.

Similarly, at $a^2 = \sqrt{2}$, each of the two bands splits into two bands. The intervals A_{ij} $(i, j = 0, 1)$ of the four bands are determined by $x_2(a^4) \leqslant \varphi_{ja^2} \circ \varphi_{ia}(x) \leqslant x_1(a^4)$ for $1 < a^2 \leqslant \sqrt{2}$.

In this manner, as a is decreased, the band splitting occurs successively at $a = \bar{a}_1, \bar{a}_2, \ldots, \bar{a}_m, \ldots$, where $\bar{a}_m = 2^{1/M}$, $M = 2^m$, and $m = 1, 2, 3, \ldots$. We put $\bar{a}_0 = 2$. For $\bar{a}_{m+1} < a < \bar{a}_m$, there exist 2^m disjoint intervals $A_{i_1 i_2 \ldots i_m}$ ($i_k = 0, 1$) in which $\rho_a(x)$ is positive (the 2^m-band regime). Here $A_{i_1 i_2 \ldots i_m}$ is the interval corresponding to

$$x_2(a_m) \leqslant \Phi_{lm}(x) \leqslant x_1(a_m) \tag{2.10}$$

where

$$\Phi_{lm} \equiv \varphi_{i_m a_{m-1}} \circ \varphi_{i_{m-1} a_{m-2}} \circ \cdots \circ \varphi_{i_2 a_1} \circ \varphi_{i_1 a} \tag{2.11}$$

$$a_k \equiv a^K \qquad (K = 2^k,\ k = 1, 2, 3, \ldots, m) \tag{2.12}$$

$$l = 1 + i_1 + 2i_2 + \cdots + 2^{m-1} i_m \tag{2.13}$$

We put $J_l \equiv A_{i_1 i_2 \ldots i_m}$ on account of (2.13), and write the Lebesgue measure of J_l as $\mu(J_l)$. Then, from (2.10), (2.11), and (2.5), we have

$$\mu(J_l) = a^{l-1}\mu(J_1) \qquad (l = 1, 2, 3, \ldots, 2^m) \tag{2.14}$$

$$\mu(J_1) = \Delta(a_m)/|\alpha_0(a)\alpha_0(a_1) \cdots \alpha_0(a_{m-1})| \tag{2.15}$$

where

$$\Delta(a) \equiv x_1(a) - x_2(a) = (a/2)(a-1) \tag{2.16}$$

Equation (2.14) implies that $f_a J_l = J_{l+1}$ for $1 \leqslant l \leqslant 2^m - 1$. We can also show that $f_a J_M = J_1$, where $M = 2^m$.

By the one-to-one transformation Φ_{lm}, we have

$$f_{a_m} = \Phi_{lm} \circ f_a^{(2^m)} \circ \Phi_{lm}^{-1} \tag{2.17}$$

Thus, for $\bar{a}_{m+1} < a \leqslant \bar{a}_m$, repeated use of (2.9) yields

$$\rho_a(x) = \sum_{l=1}^{2^m} \frac{\Delta(a_m)}{2^m \mu(J_l)} \rho_{a_m}(\Phi_{lm}(x)) \tag{2.18}$$

where $\sqrt{2} < a_m \leqslant 2$.

Similarity and the Rescaling Factor of Bandwidth

Let r be a constant such that $\sqrt{2} < r \leqslant 2$, and put $r_m = r^{1/M}$ ($M = 2^m$, $m = 0, 1, 2, \ldots$). When $a = r_m$, then $a^2 = r_{m-1}$ and $\bar{a}_{m+1} < a \leqslant \bar{a}_m$. From (2.6), we have

$$f_{r_{m-1}}^{(2^{m-1})} = \varphi_{ir_m} \circ f_{r_m}^{(2^m)} \circ \varphi_{ir_m}^{-1} \qquad (i = 0, 1) \tag{2.19}$$

This equation implies that each band for $f_{r_{m-1}}$ corresponds to two bands for

f_{r_m}, one belonging to I_0 and the other to I_1, and that the process within any band for $f_{r_{m-1}}$ generated by $f_{r_{m-1}}^{(2^{m-1})}$ is similar to the process within each of the corresponding bands for f_{r_m} generated by $f_{r_m}^{(2^m)}$.

From (2.19) and (2.5), the rescaling factors of bandwidth are given by $|\alpha_i(r_m)|$ $(i = 0, 1)$. As $m \to \infty$, they become infinity:

$$|\alpha_i(r_m)| \to 2^{m+1}/\ln r \tag{2.20}$$

For the infinite sequence $r, r_1, r_2, \ldots$, the convergence rate δ is

$$\delta = \lim_{m\to\infty} \frac{r_{m-1} - r_m}{r_m - r_{m+1}} = 2 \tag{2.21}$$

This is independent of r. If $r = 2$, then r_m is the band-splitting point $\bar{a}_m$. According to Feigenbaum,[4] $\delta = 4.669 \cdots$ for a quadratic map in its periodic region.

Invariant Density $\rho_a(x)$ for Certain Sequences of a

We give here explicit expressions of $\rho_a(x)$ for three types of sequences of values of a: the sequence of the band-splitting points $\bar{a}_m$ and the two sequences which converge to $\bar{a}_m$ from above and from below.

(I) $a = \bar{a}_m = 2^{1/M}$, where $M = 2^m$ and $m = 0, 1, 2, 3, \ldots$. If $a = \bar{a}_0 = 2$, it follows immediately from (2.1) that $\rho_a(x) = 1$ for $0 \leqslant x \leqslant 1$.[3] Since $a_m = 2$ for any m, we have $\rho_{a_m}(\Phi_{lm}(x)) = \theta_l(x)$, where θ_l is the indicator function of $J_l : \theta_l(x) = 1$ for $x \in J_l$, and $\theta_l(x) = 0$ otherwise. Hence at $a = \bar{a}_m$

$$\rho_a(x) = \sum_{l=1}^{M} \frac{1}{M\mu(J_l)} \theta_l(x) \tag{2.22}$$

(II) $a = b_{mK} \equiv b_K^{1/M}$, where $M = 2^m$, $m = 0, 1, 2, 3, \ldots$, and $K = 3, 5, 7, \ldots$. Here b_K is the value of a at which $\{x_n(a)\}$ is a periodic orbit with period K in which $x_n(a) \in (1/2, x_1(a))$ for $3 \leqslant n \leqslant K-1$, and $x_K(a) = 1/2$, i.e., a rotating periodic orbit. Then we have

$$x_n(a) = x^*(a)\left[1 + (-1)^n a^{n-3}(a-1)(a^2/2 - 1)\right] \qquad (n \geqslant 3)$$

Therefore, b_K is determined as the maximal root of the equation $s^K - 2s^{K-2} - 1 = 0$,[25] and we get $b_3 = (1+\sqrt{5})/2 > b_5 > \cdots > b_K > b_{K+2} > \cdots > \sqrt{2} = \lim_{K\to\infty} b_K$. Hence the sequence $b_{m3}, b_{m5}, b_{m7}, \ldots$ converges to $\bar{a}_{m+1}$ from above. The map f_a has a periodic orbit with period K if and only if $a \geqslant b_K$.[1,25]

At $a = b_K$, we define the intervals J_j ($j = 1, 2, \ldots, K-1$) by

$$J_j \equiv \begin{cases} [x_{j+4}(a), x_{j+2}(a)] & \text{for odd} \quad j \leqslant K-2 \\ [x_{j+2}(a), x_{j+4}(a)] & \text{for even} \quad j \leqslant K-3 \\ [x_3(a), x_4(a)] & \text{for} \quad j = K-1 \end{cases} \tag{2.23}$$

Then it follows that

$$\begin{gathered} f_a : J_1 \to J_2 \to J_3 \to \cdots \to J_{K-2} \to J_2 \cup J_4 \cup \cdots \cup J_{K-1}, \\ J_{K-1} \to J_1 \cup J_{K-1} \end{gathered} \tag{2.24}$$

At $a = b_K$, therefore, the solution to Eq. (2.1) is found to be

$$\rho_a(x) = \sum_{j=1}^{K-1} d_j(a)\theta_j(x) \tag{2.25}$$

where

$$d_j(a)/d_1(a) = \begin{cases} (a^j + 1)/a^{j-1}(a+1) & (j = 1, 3, \ldots, K-2) \\ (a^{j+1} + 1)/a^{j-1}(a+1) & (j = 2, 4, \ldots, K-3) \\ a & (j = K-1) \end{cases} \tag{2.26}$$

$$d_1(a) = 2(a+1)/a(a-1)^2[K(a^2-2)+4]$$

At $a = b_{mK}$, we have $a_m = b_K$, and $\{x_n(a)\}$ is a periodic orbit with period $2^m K$. The map f_a has a periodic orbit with period $2^m K$ if and only if $a \geqslant b_{mK}$. At $a = b_{mK}$, we must consider 2^m intervals J_{jl} for each J_j defined by (2.23):

$$J_{jl} \equiv \{x \mid \Phi_{lm}(x) \in J_j(a_m)\} \qquad (l = 1 \cdots 2^m) \tag{2.27}$$

From (2.18) and (2.25) we have

$$\rho_a(x) = \sum_{l=1}^{M} \sum_{j=1}^{K-1} \frac{\Delta(a_m)}{M\mu(J_l)} d_j(a_m)\theta_{jl}(x) \tag{2.28}$$

where θ_{jl} is the indicator function of J_{jl}, the coefficients $d_j(a_m)$ are given by (2.26), and $J_l = \bigcup_{j=1}^{K-1} J_{jl}$.

(III) $a = c_{mK} \equiv c_K^{1/M}$, where $M = 2^m$, $m = 0, 1, 2, 3, \ldots$, and $K = 3, 4, 5, 6, \ldots$. Here c_K is the value of a at which $\{x_n(a)\}$ is a periodic orbit with period K in which $x_n(a) \in (x_2(a), 1/2)$ for $3 \leqslant n \leqslant K-1$, and $x_K(a) = 1/2$, i.e., a rising periodic orbit. Then we have

$$x_n(a) = a^{n-1}(1 - a/2) \qquad (n \geqslant 2)$$

Thus c_K is the maximal root of the equation $s^K - 2s^{K-1} + 1 = 0$, and

$$c_3 = (1+\sqrt{5})/2 < c_4 < \cdots < c_K < c_{K+1} < \cdots < 2 = \lim_{K\to\infty} c_K$$

Hence the sequence $c_{m3}, c_{m4}, c_{m5}, \ldots$ converges to $\bar{a}_m$ from below.

At $a = c_K$, let us define the intervals J_j by

$$J_j = [x_{j+1}(a), x_{j+2}(a)] \qquad (j = 1, 2, \ldots, K-1) \tag{2.29}$$

Then

$$f_a : J_1 \to J_2 \to J_3 \to \cdots \to J_{K-1} \to \bigcup_{j=1}^{K-1} J_j \tag{2.30}$$

Equation (2.1) leads to $\rho_a(x)$ which takes the same form as (2.25) but with the different coefficients $d_j(a)$ being given by

$$\begin{aligned} d_j(a)/d_1(a) &= (a^j - 1)/a^{j-1}(a-1) \qquad (j = 1, 2, \ldots, K-1) \\ d_1(a) &= [a - Ka(1 - a/2)]^{-1} \end{aligned} \tag{2.31}$$

At $a = c_{mK}$, we have $a_m = c_K$, and $\{x_n(a)\}$ is a periodic orbit with period $2^m K$. Corresponding to each J_j defined by (2.29), there exist 2^m intervals

$$J_{jl} \equiv \{x \mid \Phi_{lm}(x) \in J_j(a_m)\} \qquad (l = 1 \cdots 2^m) \tag{2.32}$$

The invariant density $\rho_a(x)$ has the same form as (2.28) but with the coefficients $d_j(a_m)$ being given by (2.31).

3. CORRELATION FUNCTIONS AND POWER SPECTRA AT THE BAND-SPLITTING POINTS

The time-correlation function of orbits generated by the dynamical equation (1.2) is defined by

$$C(n) \equiv \langle f_a^{(n)}(x)x \rangle - \langle x \rangle^2 \qquad (n = 0, 1, 2, \ldots) \tag{3.1}$$

where $\langle \cdots \rangle$ is the average with $\rho_a(x)$:

$$\langle F \rangle \equiv \int_J dx\, F(x)\rho_a(x) \tag{3.2}$$

There may be a case in which $C(n)$ contains periodic components which survive at $n = \infty$. Taking this fact into account, we define the power spectrum as

$$P(\omega) = \operatorname{Re} \lim_{\epsilon\to+0} \tilde{P}(i\omega + \epsilon) \tag{3.3}$$

where

$$\tilde{P}(z) = \sum_{n=0}^{\infty} C(n)e^{-nz} \qquad (\mathrm{Re}\, z > 0) \tag{3.4}$$

The total integrated intensity of the power spectrum is given by

$$\int_0^{2\pi} d\omega\, P(\omega) = 2\pi C(0) \tag{3.5}$$

In this section, we use the formulas given in Appendix B and obtain the time-correlation function at the band-splitting point $\bar{a}_m$. The invariant density is given by (2.22). We put $N = M = 2^m$ for N in Appendix B. Then d_l in (B.1) is equal to $[M\mu(J_l)]^{-1}$.

Since $f_a J_l = J_{l+1}\ (\mathrm{mod}\, M)$ and $1/2 \in J_M$, we have

$$\mathscr{H}\theta_l(x) = \begin{cases} a^{-1}\theta_{l+1}(x) & (1 \leqslant l \leqslant M-1) \\ 2a^{-1}\theta_1(x) & (l = M) \end{cases} \tag{3.6}$$

Thus the matrix $H_0 = (\xi_{ij})$ defined by (B.3) is given by

$$\xi_{ij} = \begin{cases} a^{-1}\delta_{i,j-1} & (1 \leqslant i \leqslant M-1) \\ 2a^{-1}\delta_{1,j} & (i = M) \end{cases} \tag{3.7}$$

The eigenvalue equation for H_0 becomes

$$\det(H_0 - \lambda I) = \lambda^M - 1 = 0 \tag{3.8}$$

the eigenvalues being $\lambda_j^{(0)} = \exp[i2\pi(j-1)/M]$ $(j = 1, 2, \ldots, M)$.

The matrices $U_0 = (u_{ij})$ and $U_0^{-1} = (\bar{u}_{ij})$ are

$$u_{ij} = u_{1j} s_j^{i-1} \tag{3.9}$$

$$\bar{u}_{ij} = \bar{u}_{i1} s_i^{-(j-1)} \tag{3.10}$$

where $s_j = a\lambda_j^{(0)}$. It follows from $U_0^{-1}U_0 = I$ that

$$\bar{u}_{j1} u_{1j} = M^{-1} \qquad (1 \leqslant j \leqslant M) \tag{3.11}$$

Especially, we get

$$d_l = \bar{u}_{1l} = \bar{u}_{11} a^{-(l-1)} \tag{3.12}$$

$$\bar{u}_{11} = \left[M\mu(J_1)\right]^{-1} = M^{-1}\alpha_0(a)\alpha_0(a_1)\cdots\alpha_0(a_{m-1})$$

as they should be because of (2.22) and (2.14).

In order to obtain the matrices H_1 and H_{10} introduced in (B.9), we define the two sets of the numbers l $(= 1, 2, \ldots, M-1)$ of J_l: $\mathbf{L}_1 = \{l \mid J_l \subset [0, 1/2]\}$ and $\mathbf{L}_2 = \{l \mid J_l \subset (1/2, 1]\}$. The members of $\mathbf{L}_2$ are (i) the odd numbers or (ii) the even numbers given by $l = 2^j\,(2k-1)$, where $j = 2, 4,$

6, . . . , $m-2$ (for even $m \geqslant 4$) or $m-1$ (for odd $m \geqslant 3$), and $k = 1, 2, 3, \ldots, 2^{m-j-1}$. The members of $\mathbf{L}_1$ are the even numbers other than (ii). Then we have

$$\mathscr{H}\{x\theta_l(x)\} = \begin{cases} a^{-2}x\theta_{l+1}(x) & (l \in \mathbf{L}_1) \\ -a^{-2}x\theta_{l+1}(x) + a^{-1}\theta_{l+1}(x) & (l \in \mathbf{L}_2) \\ a^{-1}\theta_1(x) & (l = M) \end{cases} \tag{3.13}$$

Hence $H_1 = (\eta_{ij})$ and $H_{10} = (\zeta_{ij})$ are given as follows:

$$\eta_{ij} = \begin{cases} a^{-2}\delta_{i,j-1} & (i \in \mathbf{L}_1) \\ -a^{-2}\delta_{i,j-1} & (i \in \mathbf{L}_2) \\ 0 & (i = M) \end{cases} \tag{3.14}$$

$$\zeta_{ij} = \begin{cases} 0 & (i \in \mathbf{L}_1) \\ a^{-1}\delta_{i,j-1} & (i \in \mathbf{L}_2) \\ a^{-1}\delta_{1,j} & (i = M) \end{cases} \tag{3.15}$$

The eigenvalue equation for H_1 becomes

$$\det(H_1 - \lambda I) = \lambda^M = 0 \tag{3.16}$$

the eigenvalue $\lambda = 0$ being M-fold degenerate. Therefore we get for $n \geqslant M$

$$H_1^n = 0 \tag{3.17}$$

Let $\eta_{ij}^{(n)}$ be the ij-element of the matrix H_1^n, where $n = 1, 2, 3, \ldots, M-1$. Then we have from (3.14)

$$\eta_{ij}^{(n)} = \begin{cases} (-1)^{\sigma(i,i+n-1)}a^{-2n}\delta_{i,j-n} & (1 \leqslant i \leqslant M-n) \\ 0 & (M-n+1 \leqslant i \leqslant M) \end{cases} \tag{3.18}$$

where

$$\sigma(i, i+n-1) = \#\{k \mid i \leqslant k \leqslant i+n-1, k \in \mathbf{L}_2\} \tag{3.19}$$

The coupling matrix V defined by (B.12) is found to be diagonal: $V = \operatorname{diag}(v_1, v_2, \ldots, v_M)$, v_i being determined by

$$v_i = \begin{cases} a^{-1}v_{i+1} & (i \in \mathbf{L}_1) \\ -a^{-1}v_{i+1} + 1 & (i \in \mathbf{L}_2) \\ 1/2 & (i = M) \end{cases} \tag{3.20}$$

Therefore (B.14) leads to

$$J_l^{(1)} = v_l J_l^{(0)} \tag{3.21}$$

from which we see that v_l is the center of the interval J_l. It should be noted that $v_{l+1} = f_a(v_l)$ for $l = 1, 2, \ldots, M-1$, but $f_a(v_M) = x_1(a) \neq v_1$.

Using (3.9), (3.10), and (3.20), we have

$$\left(U_0^{-1} V U_0\right)_{ij} = \bar{u}_{i1} u_{1j} \sum_{l=1}^{M} v_l \left(\lambda_j^{(0)} / \lambda_i^{(0)}\right)^{l-1} \tag{3.22}$$

From (B.16), (3.22), and (3.11), the average $\langle x \rangle$ is obtained:

$$\langle x \rangle = M^{-1} \sum_{l=1}^{M} v_l \tag{3.23}$$

The correlation function consists of two parts:

$$C(n) = G_0(n) + G_1(n) \tag{3.24}$$

where $G_0(n)$ is given by (B.18), and $G_1(n)$ by (B.19). At $a = \bar{a}_m$, the function $G_0(n)$ is periodic with period M as seen from the eigenvalues $\lambda_j^{(0)}$. We put

$$\lambda_j^{(0)} = \exp(i\omega_j), \qquad \omega_j = 2\pi(j-1)/M \qquad (j = 1, 2, \ldots, M) \tag{3.25}$$

Then we obtain from (B.18), (3.22), and (3.11)

$$G_0(n) = \sum_{j=2}^{M} |A(\omega_j)|^2 \exp(in\omega_j) \qquad (n = 0, 1, 2, \ldots) \tag{3.26}$$

where

$$A(\omega_j) = M^{-1} \sum_{l=1}^{M} v_l \exp\left[i(l-1)\omega_j\right] \tag{3.27}$$

The function $G_1(n)$ may be called the stochastic part, the randomness of which comes from the distribution of J_l in the interval J and the mixing property within each band. If we put

$$\mathbf{J}' \equiv \mathbf{J}^{(2)} - V\mathbf{J}^{(1)} = \int_J x \boldsymbol{\theta}^{(1)}(x)\, dx \tag{3.28}$$

where $\boldsymbol{\theta}^{(1)}$ is defined by (B.11), then the elements J_l' of the vector $\mathbf{J}'$ satisfy the relation

$$J_l' = a^{-3(M-l)} J_M' \qquad (1 \leqslant l \leqslant M) \tag{3.29}$$

In deriving this relation, we have applied (A.3) and (B.14) to the fact that

$$\mathscr{H}\left\{x\theta_l^{(1)}(x)\right\} = \begin{cases} a^{-3} x \theta_{l+1}^{(1)}(x) & (l \in \mathbf{L}_1) \\ a^{-3} x \theta_{l+1}^{(1)}(x) - a^{-2} \theta_{l+1}^{(1)}(x) & (l \in \mathbf{L}_2) \end{cases} \tag{3.30}$$

which is obtained with the aid of (3.13) and (3.20). Since

$$J_M = \{[1 - \mu(J_M)]/2, [1 + \mu(J_M)]/2\}$$

at $a = \bar{a}_m$, J'_M is given by

$$J'_M = [\mu(J_M)]^3/12 \tag{3.31}$$

Consequently we obtain

$$G_1(n) = \begin{cases} (16M)^{-1}[\mu(J_M)]^2/(1 - a^{-2}) & (n = 0) \\ (48M)^{-1}[\mu(J_M)]^2 a^n \sum_{l=1}^{M-n} (-1)^{\sigma(l,l+n-1)} a^{2l}, & \\ & (n = 1, 2, \ldots, M-1) \\ 0 & (n \geqslant M) \end{cases} \tag{3.32}$$

where use has been made of (B.19), (3.12), (3.17), (3.18), (3.29), and (3.31). It should be noted that $|G_1(n)| < G_1(0)$ for $n > 0$. If $n = 0$ in (3.24), it follows that

$$C(0) = \langle x^2 \rangle - \langle x \rangle^2 = G_0(0) + G_1(0) \tag{3.33}$$

From (3.24), the power spectrum is written as

$$P(\omega) = P_0(\omega) + P_1(\omega) \tag{3.34}$$

For the periodic part we have from (3.26)

$$P_0(\omega) = \frac{1}{2} G_0(0) + \pi \sum_{j=2}^{M} |A(\omega_j)|^2 \delta(\omega - \omega_j) \tag{3.35}$$

for $0 \leqslant \omega < 2\pi$, and

$$\int_0^{2\pi} d\omega\, P_0(\omega) = 2\pi G_0(0) = 2\pi \sum_{j=2}^{M} |A(\omega_j)|^2 \tag{3.36}$$

The stochastic part is a continuous spectrum given by

$$P_1(\omega) = \sum_{n=0}^{M-1} G_1(n) \cos(n\omega) \tag{3.37}$$

the total intensity being

$$\int_0^{2\pi} d\omega\, P_1(\omega) = 2\pi G_1(0) = 2\pi (16M)^{-1}[\mu(J_M)]^2/(1 - a^{-2}) \tag{3.38}$$

In particular, we have $G_0(n) = 0$ and $G_1(n) = (1/12)\delta_{n,0}$ at $a = \bar{a}_0 = 2$. At $a = \bar{a}_1 = \sqrt{2}$, Eqs. (3.26) and (3.32) lead to the expression for $C(n)/C(0)$ obtained in the previous paper.[1] Figure 2 shows the power spectra at the band-splitting points $\bar{a}_m$ for $m = 1, 2, 3, 4, 5, 6$.

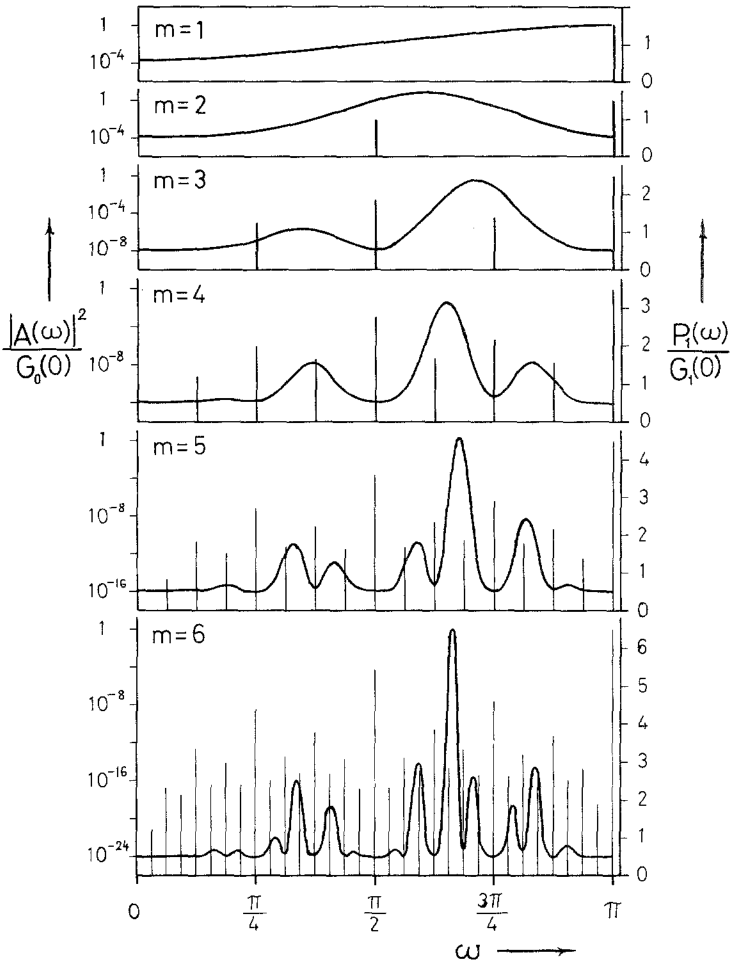

Fig. 2. Power spectra of the tent map at the band-splitting points $\bar{a}_m$. The relative intensity $|A(\omega)|^2/G_0(0)$ of the purely periodic modes [line spectrum, see (3.35)] and the relative spectrum $P_1(\omega)/G_1(0)$ of the stochastic part [continuous spectrum, see (3.37)] are shown as a function of ω for $m = 1, 2, 3, 4, 5, 6$.

4. CORRELATION FUNCTIONS AT $a = b_{mK}$ AND AT $a = c_{mK}$

At $a = b_{mK}$ ($K = 3, 5, 7, \ldots,\ m = 0, 1, 2, 3, \ldots$), the invariant density is found to be (2.28). Since in the single-band regime ($m = 0$, $a = b_K$) the flow of $K - 1$ intervals J_j generated by f_a is given by (2.24), the flow of N $[= 2^m(K-1)]$ intervals J_{jl}, (2.27), generated by f_a in the corresponding

2^m-band regime ($a = b_{mK}$) becomes

$$
\begin{aligned}
f_a : J_{jl} &\to J_{j,l+1} && (l = 1 \cdots M-1, j = 1 \cdots K-1) \\
J_{jl} &\to J_{j+1,1} && (l = M, j = 1 \cdots K-3) \\
J_{jl} &\to J_{21} \cup J_{41} \cup \cdots \cup J_{K-1,1} && (l = M, j = K-2) \\
J_{jl} &\to J_{11} \cup J_{K-1,1} && (l = M, j = K-1)
\end{aligned}
\tag{4.1}
$$

where $M = 2^m$. Therefore we have for $H_0 = (\xi_{jl,j'l'})$ and $H_1 = (\eta_{jl,j'l'})$ defined in Appendix B,

$$
\xi_{jl,j'l'} = \begin{cases} a^{-1}\delta_{j,j'}\delta_{l,l'-1} & (l = 1 \cdots M-1, j = 1 \cdots K-1) \\ a^{-1}\delta_{j,j'-1}\delta_{1,l'} & (l = M, j = 1 \cdots K-3) \\ a^{-1}(\delta_{2,j'} + \delta_{4,j'} + \cdots + \delta_{K-1,j'})\delta_{1,l'} & (l = M, j = K-2) \\ a^{-1}(\delta_{1,j'} + \delta_{K-1,j'})\delta_{1,l'} & (l = M, j = K-1) \end{cases}
\tag{4.2}
$$

and

$$
\eta_{jl,j'l'} = \begin{cases} \epsilon_l a^{-2}\delta_{j,j'}\delta_{l,l'-1} & (l = 1 \cdots M-1, j = 1 \cdots K-1) \\ (-1)^{m+1}a^{-2}\delta_{j,j'-1}\delta_{1,l'} & (l = M, j = 1 \cdots K-3) \\ (-1)^m a^{-2}(\delta_{2,j'} + \delta_{4,j'} + \cdots + \delta_{K-1,j'})\delta_{1,l'} & (l = M, j = K-2) \\ (-1)^{m+1}a^{-2}(\delta_{1,j'} + \delta_{K-1,j'})\delta_{1,l'} & (l = M, j = K-1) \end{cases}
\tag{4.3}
$$

where

$$
\epsilon_l = \begin{cases} 1 & (l \in \mathbf{L}_1) \\ -1 & (l \in \mathbf{L}_2) \end{cases}
\tag{4.4}
$$

The eigenvalues $\lambda_{jl}^{(0)}$ are determined from the equation

$$
\det(H_0 - \lambda I) = a^{-N}(S^K - 2S^{K-2} - 1)/(S+1) = 0 \tag{4.5}
$$

and the eigenvalues $\lambda_{jl}^{(1)}$ from the equation

$$
\det(H_1 - \lambda I) = a^{-2N}(T^K - 1)/(T-1) = 0, \qquad N \equiv M(K-1) \tag{4.6}
$$

Here $S = s^M$, $s = a\lambda$, and $T = t^M$, $t = a^2\lambda$, where $M = 2^m$. In deriving (4.6), the fact that $\epsilon_1\epsilon_2 \cdots \epsilon_{M-1} = (-1)^m$ has been used.

Let $s_j = |s_j|\exp(i\varphi_j)$ ($j = 1 \cdots K-1$) be the $K-1$ roots of Eq. (4.5) in the single-band regime ($M = 1$). Then the N roots s_{jl} ($j = 1 \cdots K-1$,

$l = 1 \cdots M$) of Eq. (4.5) can be written as

$$s_{jl} = |s_j|^{1/M} \exp\{ i[2\pi(l-1) + \varphi_j]/M \} \tag{4.7}$$

Note that $s_{11} = a\lambda_{11}^{(0)} = a = b_{mK}$. The N roots t_{jl} of Eq. (4.6) are

$$t_{jl} = \exp\{ i2\pi(l - 1 + j/K)/M \} \tag{4.8}$$

Since the eigenvalues for H_0 and H_1 are different from each other, the formula (B.23) can be used, and the correlation function at $a = b_{mK}$ takes the form

$$\begin{aligned} \frac{C(n)}{C(0)} &= \sum_{l=2}^{M} A_{1l}^{(0)} \left(\frac{s_{1l}}{a} \right)^n + \sum_{j=2}^{K-1} \sum_{l=1}^{M} A_{jl}^{(0)} \left(\frac{s_{jl}}{a} \right)^n \\ &\quad + \sum_{j=1}^{K-1} \sum_{l=1}^{M} A_{jl}^{(1)} \left(\frac{t_{jl}}{a^2} \right)^n \end{aligned} \tag{4.9}$$

The coefficients $A_{jl}^{(0)}$ and $A_{jl}^{(1)}$ can be obtained straightforwardly through (B.25)–(B.29) in terms of s_{jl} and t_{jl}, respectively. They are, however, somewhat lengthy, and here we do not present them. In the single-band regime ($m = 0$, $a = b_K$), they are in agreement with those obtained in the previous paper.[1]

In the single-band regime, Eq. (4.5) has two real roots $s_1 = b_K > \sqrt{2}$ and $s_2 = -b'_K$, where $1 < b'_K < \sqrt{2}$. Other $K - 3$ roots are complex and satisfy the condition $3^{-1/(K-2)} < |s_j| < 1$ ($j = 3 \cdots K - 1$). Therefore, at $a = b_{mK}$, we have $s_{1l} = a \exp(i\omega_l)$, where $\omega_l = 2\pi(l-1)/M$, and $|s_{jl}| < a$ for s_{jl} other than s_{1l}. The first term on the right-hand side of (4.9) represents $2^m - 1$ periodic modes, and both the second and the third term decay as $n \to \infty$. In the power spectrum they give δ-function peaks at $\omega = \omega_l$ and a continuous spectrum, respectively.

The correlation function at $a = c_{mK}$ ($K = 3, 4, 5, 6, \ldots, m = 0, 1, 2, 3, \ldots.$) is obtained similarly. In view of (2.30) the flow of N [$= 2^m(K-1)$] intervals defined by (2.32) is found to be

$$\begin{aligned} f_a : J_{jl} &\to J_{j,l+1} \qquad (l = 1 \cdots M-1, j = 1 \cdots K-1) \\ J_{jl} &\to J_{j+1,1} \qquad (l = M, j = 1 \cdots K-2) \\ J_{jl} &\to \bigcup_{i=1}^{K-1} J_{i1} \qquad (l = M, j = K-1) \end{aligned} \tag{4.10}$$

where $M = 2^m$. The matrices $H_0 = (\xi_{jl,j'l'})$ and $H_1 = (\eta_{jl,j'l'})$ are

$$\xi_{jl,j'l'} = \begin{cases} a^{-1}\delta_{j,j'}\delta_{l,l'-1} & (l = 1 \cdots M-1, j = 1 \cdots K-1) \\ a^{-1}\delta_{j,j'-1}\delta_{1,l'} & (l = M, j = 1 \cdots K-2) \\ a^{-1}\delta_{1,l'} & (l = M, j = K-1) \end{cases} \tag{4.11}$$

and

$$\eta_{jl,j'l'} = \begin{cases} \epsilon_l a^{-2}\delta_{j,j'}\delta_{l,l'-1} & (l = 1 \cdots M-1, j = 1 \cdots K-1) \\ (-1)^m a^{-2}\delta_{j,j'-1}\delta_{1,l'} & (l = M, j = 1 \cdots K-2) \\ (-1)^{m+1} a^{-2}\delta_{1,l'} & (l = M, j = K-1) \end{cases} \tag{4.12}$$

The eigenvalue equations are

$$\det(H_0 - \lambda I) = (-1)^N a^{-N}(S^K - 2S^{K-1} + 1)/(S-1) = 0 \tag{4.13}$$

and

$$\det(H_1 - \lambda I) = (-1)^N a^{-2N}(T^K - 1)/(T-1) = 0 \tag{4.14}$$

where, as in (4.5) and (4.6), $S = s^M$, $s = a\lambda$, and $T = t^M$, $t = a^2\lambda$. The roots of Eq. (4.13) are also written as (4.7), in which s_j are the roots of Eq. (4.13) in the single-band regime. Note that $s_{11} = a\lambda_{11}^{(0)} = a = c_{mK}$. The roots of Eq. (4.14) are given by (4.8).

The correlation function at $a = c_{mK}$ also takes the form (4.9). We omit here writing down the explicit expressions for the coefficients $A_{jl}^{(0)}$ and $A_{jl}^{(1)}$. In the two-band regime ($m = 1, a = \sqrt{c_K}$), they agree with those obtained in the previous paper.[1]

For the roots of Eq. (4.13) in the single-band regime ($m = 0$, $a = c_K$), we have $s_1 = c_K \geqslant (1+\sqrt{5})/2$, and $3^{-1/(K-1)} < |s_j| < 1$ for $j = 2 \cdots K-1$. At $a = c_{mK}$, therefore, we have $s_{1l} = a\exp(i\omega_l)$, where $\omega_l = 2\pi(l-1)/M$, and $|s_{jl}| < 1$ for s_{jl} other than s_{1l}. Also in this case, the first term on the right-hand side of (4.9) represents periodic modes, and both the second and the third term tend to zero as $n \to \infty$.

5. CRITICAL BEHAVIORS NEAR THE BAND-SPLITTING POINTS

Extending the previous treatment,[1] we consider the critical behavior of the correlation function when the parameter a approaches the band-splitting point $\bar{a}_m$ ($m = 1, 2, 3, \ldots$) according to the sequence $b_{m-1,K}$ ($K = 3, 5, 7, \ldots$): $b_{m-1,K} \downarrow \bar{a}_m$ as $K \to \infty$.

As $K \to \infty$, the two real roots of Eq. (4.5) in the single-band regime behave as

$$s_1 = b_K = \sqrt{2} + 2^{-(K+1)/2} + \cdots \tag{5.1}$$

and

$$s_2 = -b'_K, \qquad b'_K = \sqrt{2} - 2^{-(K+1)/2} + \cdots \tag{5.2}$$

For the other $K-3$ roots, we have $|s_j| \to 1 - 0$ ($j = 3 \cdots K-1$).

When $a = b_{m-1,K}$ and K tends to infinity, therefore, the damping becomes small for the 2^{m-1} modes with the eigenvalues $s_{2l} = (b'_K)^{2/M} \exp[i2\pi(2l-1)/M]$ $(l = 1, 2, \ldots, M/2)$, where $M = 2^m$. The damping of other modes, including the modes with the eigenvalues t_{jl}, remains finite. Indeed, at $a = b_{m-1,K}$ for large K, we have

$$(s_{2l}/a)^n = \exp(-n\gamma_K/2^{m-1})\exp[i2\pi(2l-1)n/2^m] \qquad (l = 1 \cdots 2^{m-1}) \tag{5.3}$$

where γ_K is the damping constant in the single-band regime:[1]

$$\gamma_K \equiv \ln(b_K/b'_K) = \sqrt{2}\left(b_K - \sqrt{2}\right) + \cdots = 2^{-K/2} + \cdots \tag{5.4}$$

These 2^{m-1} modes are the critical modes which presage the splitting of the 2^{m-1} bands into the 2^m bands. The damping constant of these modes is

$$\gamma_{m-1,K} = \gamma_K/2^{m-1} = (2/\bar{a}_m)(b_{m-1,K} - \bar{a}_m) + \cdots \tag{5.5}$$

This is independent of their frequencies $\omega = 2\pi(2l-1)/2^m$. The power spectrum corresponding to these modes consists of 2^{m-1} peaks at $\omega = 2\pi(2l-1)/2^m$, the line shape near the respective peaks being Lorentzian with the width $\gamma_{m-1,K}$.

For the $2^{m-1}-1$ periodic modes, we have

$$(s_{1l}/a)^n = \exp[i2\pi(l-1)n/2^{m-1}] \qquad (l = 2 \cdots 2^{m-1}) \tag{5.6}$$

They show $2^{m-1}-1$ δ-function peaks at $\omega = 2\pi(l-1)/2^{m-1}$ in the power spectrum.

As $K \to \infty$, the part of the correlation function (4.9), where m is replaced by $m-1$,

$$\sum_{l=2}^{2^{m-1}} A_{1l}^{(0)}\left(\frac{s_{1l}}{a}\right)^n + \sum_{l=1}^{2^{m-1}} A_{2l}^{(0)}\left(\frac{s_{2l}}{a}\right)^n \tag{5.7}$$

tends to $G_0(n)/C(0)$ at $a = \bar{a}_m$ given by (3.26) and (3.33), and the sum of the remaining terms tends to $G_1(n)/C(0)$ at $a = \bar{a}_m$ given by (3.32) and (3.33).

We next consider the case in which the band-splitting point $\bar{a}_m$ is approached from below according to the sequence c_{mK} $(K = 3, 4, 5, 6, \ldots)$: $c_{mK} \uparrow \bar{a}_m$ as $K \to \infty$. As mentioned in Section 4, the $2^m - 1$ modes with the eigenvalues s_{1l} have no damping. We have

$$(s_{1l}/a)^n = \exp[i2\pi(l-1)n/2^m] \qquad (l = 2, 3, \ldots, 2^m) \tag{5.8}$$

which give $2^m - 1$ δ-function peaks at $\omega = 2\pi(l-1)/2^m$ in the power spectrum. As $K \to \infty$, the first term of the correlation function (4.9),

$$\sum_{l=2}^{2^m} A_{1l}^{(0)}(s_{1l}/a)^n \tag{5.9}$$

tends to $G_0(n)/C(0)$ at $a = \bar{a}_m$, and the sum of all the terms except this one tends to $G_1(n)/C(0)$ at $a = \bar{a}_m$.

6. CRITICAL BEHAVIORS NEAR THE CHAOTIC TRANSITION POINT

We now investigate the critical behavior of the power spectrum as the transition point $a = 1$ to the nonchaotic state is approached from the chaotic side. We put $a = \bar{a}_m$ and let m be large. Then we have

$$\eta \equiv a - 1 = \epsilon + \epsilon^2/2 + \cdots, \qquad \epsilon = 2^{-m}\ln 2 \tag{6.1}$$

At $a = \bar{a}_m$, we have for v_l of (3.23) and (3.27)

$$x_3(a) < v_{2l-1} < x_1(a) \quad \text{and} \quad x_2(a) < v_{2l} < x_4(a) \tag{6.2}$$

for $l = 1, 2, 3, \ldots, M/2$, where $M = 2^m$. Here $x_n(a)$ is defined by (2.4), and use has been made of the fact that v_l is the center of the interval J_l. Since $x_1(a) - x_3(a) = (\eta^2/2)(1 + \eta)$ and $x_4(a) - x_2(a) = (\eta^2/2)(1 + 2\eta + \eta^2)$, we get $v_{2l-1} = (1 + \eta)/2$ and $v_{2l} = 1/2$ if the terms of $O(\eta^2)$ are neglected.

In order to obtain the higher-order terms, we may use the fact that

$$\begin{aligned} &x_5(a) < v_{4l-3} < x_1(a), \qquad x_2(a) < v_{4l-2} < x_6(a) \\ &x_3(a) < v_{4l-1} < x_7(a), \qquad x_8(a) < v_{4l} < x_4(a) \end{aligned} \tag{6.3}$$

for $l = 1, 2, 3, \ldots, M/4$, and $|x_j(a) - x_{j+4}(a)| = \eta^3 + O(\eta^4)$ for $j = 1, 2, 3, 4$, and that

$$\begin{aligned} &x_9(a) < v_{8l-7} < x_1(a), \qquad x_2(a) < v_{8l-6} < x_{10}(a) \\ &x_3(a) < v_{8l-5} < x_{11}(a), \qquad x_{12}(a) < v_{8l-4} < x_4(a) \\ &x_5(a) < v_{8l-3} < x_{13}(a), \qquad x_{14}(a) < v_{8l-2} < x_6(a) \\ &x_{15}(a) < v_{8l-1} < x_7(a), \qquad x_8(a) < v_{8l} < x_{16}(a) \end{aligned} \tag{6.4}$$

for $l = 1, 2, 3, \ldots, M/8$, and $|x_j(a) - x_{j+8}(a)| = 4\eta^4 + O(\eta^5)$ for $j = 1, 2, \ldots, 8$, and so on.

If we neglect the terms of $O(\eta^4)$, we have from (6.4)

$$\begin{aligned} &v_{8l-7} = (1 + \eta)/2, \qquad && v_{8l-6} = (1 - \eta^2)/2 \\ &v_{8l-5} = (1 + \eta - \eta^2 - \eta^3)/2, && v_{8l-4} = (1 + 2\eta^3)/2 \\ &v_{8l-3} = (1 + \eta - 2\eta^3)/2, && v_{8l-2} = (1 - \eta^2 + 2\eta^3)/2 \\ &v_{8l-1} = (1 + \eta - \eta^2 + \eta^3)/2, && v_{8l} = 1/2 \end{aligned} \tag{6.5}$$

Hence we obtain from (3.23), (3.27), and (3.26)

$$\langle x\rangle = (4 + 2\eta - 2\eta^2 + \eta^3)/8 \tag{6.6}$$

$$\begin{aligned} A(\pi) &= (\eta/4)(1 - 3\eta^2/2) \\ A(\pi/2) &= A^*(3\pi/2) = (\eta^2/8)(1 - i - \eta) \\ A(\pi/4) &= A^*(7\pi/4) = -(\eta^3/8)(\sqrt{2} - 1 + i) \\ A(3\pi/4) &= A^*(5\pi/4) = (\eta^3/8)(\sqrt{2} + 1 + i) \end{aligned} \tag{6.7}$$

and

$$G_0(0) = (\eta/4)^2(1 - 2\eta^2) \tag{6.8}$$

The coefficients $A(\omega_j)$ other than those given in (6.7) are of higher order. Generally, we have $A(\pi/2^{k-1}) \sim \eta^k$ for $k = 1, 2, 3, \ldots$.

Let us write the value of a explicitly in $G_0(0)$ and $A(\omega_j)$. We consider the average intensity at $\bar{a}_m$ of the periodic modes which become periodic at $\bar{a}_k$ and remain periodic for $a < \bar{a}_k$:

$$\phi(k; \bar{a}_m) \equiv \frac{1}{2^{k-1}} \sum_{l=1}^{2^{k-1}} \left| A\left(\frac{2l-1}{2^{k-1}} \pi; \bar{a}_m \right) \right|^2 \qquad (k \leqslant m) \tag{6.9}$$

Then we have, from (6.7), (6.8), (6.1), and (2.20), the scaling relations

$$G_0(0; \bar{a}_m)/G_0(0; \bar{a}_{m-1}) = \phi(1; \bar{a}_m)/\phi(1; \bar{a}_{m-1}) = 1/4 \tag{6.10}$$

$$\phi(k; \bar{a}_m)/\phi(k; \bar{a}_{m-1}) = (1/4)^k \qquad (k = 1, 2, 3) \tag{6.11}$$

$$\begin{aligned} \phi(1; \bar{a}_m)/\phi(2; \bar{a}_m) &= 2/\eta^2 = [\alpha_i(\bar{a}_m)]^2/2 \\ \phi(2; \bar{a}_m)/\phi(3; \bar{a}_m) &= 1/2\eta^2 = [\alpha_i(\bar{a}_m)]^2/8 \end{aligned} \tag{6.12}$$

for large m, where $\alpha_i(\bar{a}_m)$ is the rescaling factor defined in (2.5).

Equations (6.7) and (6.8) state that, as the transition point is approached ($a \to 1$), the total intensity (3.36) of the periodic spectrum becomes small as $(a - 1)^2$, and only the motion of period 2 is dominant. The relative intensity $|A(\pi)|^2/G_0(0)$ of the motion of period 2 evaluated from (3.27) and (3.26) is as follows: $|A(\pi)|^2/G_0(0) = 1.0$ for $m = 1$, 0.98383 (0.96144) for $m = 2$, 0.99277 (0.99167) for $m = 3$, 0.99814 (0.99803) for $m = 4$, 0.99953 (0.99952) for $m = 5$, 0.99988 (0.99988) for $m = 6$. The values in parentheses are those obtained from the asymptotic expressions (6.7) and (6.8).

We next consider the stochastic part. Let W_m be the root-mean-square bandwidth. At $a = \bar{a}_m$, we have from (2.14)

$$W_m^2 \equiv \frac{1}{M} \sum_{l=1}^{M} [\mu(J_l)]^2 = \frac{3}{4M} \frac{a^2}{a^2 - 1} [\mu(J_M)]^2 \tag{6.13}$$

and therefore, from (3.32),

$$G_1(0; \bar{a}_m) = W_m^2/12 \tag{6.14}$$

where the value of a is explicitly written in $G_1(0)$. Thus we have

$$\frac{G_1(0; \bar{a}_m)}{G_1(0; \bar{a}_{m-1})} = \frac{W_m^2}{W_{m-1}^2} = \frac{\bar{a}_{m-1} + 1}{2\bar{a}_{m-1}} \cdot \frac{1}{\left[\alpha_1(\bar{a}_m)\right]^2} \to \frac{(\ln 2)^2}{2^{2m+2}} \tag{6.15}$$

as $m \to \infty$, where $\alpha_1(\bar{a}_m)$ is the rescaling factor defined in (2.5). It may be noted that $G_1(0; \bar{a}_0) = 1/12$, where $\bar{a}_0 = 2$. As $a \to 1$, the total intensity (3.38) of the stochastic part vanishes faster than any power of $(a - 1)$.

For the ratio between the two parts, numerical evaluation from (3.36) and (3.38) leads to the following results: $G_1(0)/G_0(0) = 3.43 \times 10^{-1}$ for $m = 1$, 5.55×10^{-3} for $m = 2$, 4.01×10^{-5} for $m = 3$, 7.45×10^{-8} for $m = 4$, 3.49×10^{-11} for $m = 5$, 4.09×10^{-15} for $m = 6$. From (2.14) we have

$$\mu(J_M) = \prod_{j=1}^{m} (\bar{a}_j - 1)/(\bar{a}_j + 1) < 2^{-m(m+3)/2} \tag{6.16}$$

at $a = \bar{a}_m$. Using (6.16), (3.32), and (6.8), we have

$$G_1(0; \bar{a}_m)/G_0(0; \bar{a}_m) < (\ln 2)^{-3} 2^{-m(m+1)-1} \tag{6.17}$$

for any m.

7. SUMMARY AND CONCLUDING REMARKS

For the chaotic region ($1 < a \leqslant 2$) of the tent map (1.1), the invariant density $\rho_a(x)$, the time-correlation function of orbits $C(n)$ and its power spectrum $P(\omega)$ have been calculated exactly, and the band structure and the critical behaviors have been studied in detail.

If $a = 2$, then $\rho_a(x) = 1$, $C(n) = (1/12)\delta_{n,0}$ (δ-correlated), and $P(\omega) = 1/12$ (white). As a is decreased, a sequence of band-splitting transitions occurs at $a = \bar{a}_m$ ($m = 1, 2, 3, \ldots$), and the accumulation point of $\bar{a}_m$ is the transition point $a = 1$ to the nonchaotic region.

When the 2^{m-1}-band regime changes to the 2^m-band regime, the 2^{m-1} modes with the frequencies $\omega = 2\pi(2l - 1)/2^m$ ($l = 1 \cdots 2^{m-1}$) undergo the critical slowing-down. Their damping constant is given by $\gamma_{m-1} = (2/\bar{a}_m)(a - \bar{a}_m)$ as shown in (5.5), the inverse of which is exactly equal to the average hopping time for the band-to-band hopping process slightly above $\bar{a}_m$.[2] For $a \leqslant \bar{a}_m$ these 2^{m-1} modes become periodic modes without

[2] The idea of the average hopping time was introduced by Yorke and Yorke,[26] who discussed a decay of metastable chaos. Shenker and Kadanoff[27] applied this idea to band-splitting transitions.

damping. In the 2^{m-1}-band regime ($\bar{a}_m < a \leqslant \bar{a}_{m-1}$), there are already $2^{m-1}-1$ periodic modes with the frequencies $\omega = 2\pi(l-1)/2^{m-1}$ ($l = 2 \cdots 2^{m-1}$), and these modes remain periodic for $a \leqslant \bar{a}_m$. After all there are $2^m - 1$ periodic modes in the 2^m-band regime ($\bar{a}_{m+1} < a \leqslant \bar{a}_m$). In this sense the order in chaos increases stepwise as a band-splitting point is passed. However, it should be emphasized that the Lyapunov exponent (1.3) varies smoothly without any singularity. Both the Kolmogorov–Sinai entropy and the topological entropy are also given by $\ln a$ for the present tent map.[25] Therefore these quantities also exhibit no singularity at the band-splitting points.

In the single-band regime there is only one critical mode, the frequency of which is π and the damping is given by (5.4). Numerical calculations presented in the previous paper[1] have clarified that this mode contributes dominantly to the power spectrum in the single-band regime near $\bar{a}_1$. Since $G_1(0)/G_0(0) \ll 1$ for large m as shown in (6.17), the 2^{m-1} critical modes and the $2^{m-1}-1$ periodic modes stated above, namely, the terms given by (5.7), make also a dominant contribution to the power spectrum in the 2^{m-1}-band regime. Furthermore, (6.7) and (6.8) show that the mode with the frequency π is most dominant for large m, that is, near the chaotic transition point $a = 1$.

The total intensity of the periodic part of the power spectrum at the band-splitting point $\bar{a}_m$ behaves as $G_0(0; \bar{a}_m) = (a-1)^2/16$ near the chaotic transition point and satisfies the scaling relation (6.10) for large m. The same are true of the total intensity of the power spectrum $C(0; \bar{a}_m) = G_0(0; \bar{a}_m) + G_1(0; \bar{a}_m)$, because the intensity of the stochastic part $G_1(0; \bar{a}_m)$ is much smaller than $G_0(0; \bar{a}_m)$ as shown in (6.17). It should be noted that these power law and scaling relation are caused by the oscillatory motion of period 2, in spite of being in the chaotic region where the Lyapunov exponent is positive.

We now make remarks on some scaling laws obtained for maps which have a quadratic maximum and therefore exhibit period-doubling bifurcations to chaos. Let z be the exponent specifying the nature of the maximum: $z = 1$ for the tent map, and $z = 2$ for maps with a quadratic maximum.[27] Let us denote the control parameter of the maps by a, the band-splitting points by $\bar{a}_m$, and the chaotic transition point by a_c. We consider the scaling laws which hold in the chaotic region near the chaotic threshold a_c.

Huberman and Rudnick[19] found that the Lyapunov exponent behaves as $\lambda = \lambda_0(a - a_c)^\tau$, where τ is a universal exponent given by $\tau = \ln 2/\ln \delta$ with $\delta = 4.669 \cdots$ being Feigenbaum's convergence rate.[4] For the present tent map, (1.3) leads to $\lambda = a - 1$ near $a_c = 1$. Since $\delta = 2$ as shown in (2.21), the above power law holds for the tent map, in spite of the fact that the tend map does not exhibit period-doubling bifurcations to chaos. A sequence of band-splitting transitions is essential for this power law. However, τ takes a different value:

$$\tau = \frac{\ln 2}{\ln \delta} = \begin{cases} 1 & \text{for} \quad z = 1 \\ 0.4498\ldots & \text{for} \quad z = 2 \end{cases} \tag{7.1}$$

Nauenberg and Rudnick[22] showed that the ratio $\phi(k)/\phi(k+1)$ approaches a universal constant $2\beta^{(2)} = 20.963\ldots$ for large k, provided $k \ll m$, where $\phi(k)$ is the average intensity of the δ-function peaks in the power spectrum. Their numerical results suggest that $\phi(k)/\phi(k+1) = 2\beta^{(2)}$ holds also for small k in a good approximation if m is large enough.[22] In the approximation we have $G_0(0;\bar{a}_m)/G_0(0;\bar{a}_{m-1}) = \phi(1;\bar{a}_m)/\phi(1;\bar{a}_{m-1})$ for the two maps they considered, provided that m is large enough,[3] where $G_0(0,\bar{a}_m)$ and $\phi(1;\bar{a}_m)$ are the total intensity of the periodic part of the spectrum and the intensity of the motion of period 2, respectively. If we assume that Feigenbaum's argument[5] may be applied also to the chaotic region, we get $\phi(1;\bar{a}_m)/\phi(1,\bar{a}_{m-1}) \simeq 1$. Comparing this with (6.10), we have

$$\xi \equiv \frac{G_0(0;\bar{a}_m)}{G_0(0;\bar{a}_{m-1})} = \frac{\phi(1;\bar{a}_m)}{\phi(1;\bar{a}_{m-1})} \begin{cases} = 1/4 & \text{for} \quad z = 1 \\ \simeq 1 & \text{for} \quad z = 2 \end{cases} \tag{7.2}$$

The scaling relations (6.11) and (6.12) are also different from theirs. This is owing to the divergence of the rescaling factors (2.20) as $m \to \infty$.

This same reason applies to another scaling behavior. According to Huberman and Zisook,[21] there exists the scaling relation for the average bandwidth: $W_m = W_0 \beta^{-m}$ for large m, where $\beta = 3.2375\cdots$ is a universal constant. They showed that $G_1(0;\bar{a}_m)/G_1(0;\bar{a}_{m-1}) = W_m^2/W_{m-1}^2 = \beta^{-2}$ for large m, where $G_1(0;\bar{a}_m)$ is the total intensity of the stochastic part of the spectrum and corresponds to $N(\tilde{r}_m)$ in their notation. This relation should be compared with (6.15) for the tent map. We have for large m

$$\frac{G_1(0;\bar{a}_m)}{G_1(0;\bar{a}_{m-1})} = \frac{W_m^2}{W_{m-1}^2} = \begin{cases} [\alpha_1(\bar{a}_m)]^{-2} & \text{for} \quad z = 1 \\ \beta^{-2} & \text{for} \quad z = 2 \end{cases} \tag{7.3}$$

Their scaling relation leads to the power law $G_1(0;a) = \text{const} \times (a - a_c)^\sigma$ near the chaotic threshold, with σ a universal exponent given by $\sigma = 2\ln\beta/\ln\delta = 1.5247\cdots$. For the tent map $G_1(0;a)$ vanishes faster than any power of $(a-1)$.

Thus the difference of the "universal" constants is quite evident between the two types of maps; $z = 1$ and $z = 2$. It is expected to study further the dependence of the scaling laws on the exponent z. For this

[3] It should be noted that the present $\phi(k;\bar{a}_m)$, defined by (6.9), corresponds to their $\phi(k-1)$.

purpose, it may be necessary to consider the difference in structure of attractors near the chaotic transition point.

The present method for the calculation of correlation functions can be easily applied to maps which are piecewise linear and continuous. It is interesting to extend the method so as to treat maps with $z \neq 1$. The discussion on the band structure given in Section 2 is also expected to be extended. As an example, we may take the logistic model. This model is much complicated: as the height of the maximum is lowered, there appear infinitely many windows in each of which a unique cycle of period $K(\geqslant 3)$ or one of its subharmonics is attractive.[2] The tent map has no such windows. However, the present method and discussions may be extended at least to the band-splitting points at which there exists an absolutely-continuous invariant measure.

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APPENDIX A. SOME PROPERTIES OF THE FROBENIUS–PERRON OPERATOR

For a map f on the interval J, the Frobenius–Perron operator $\mathscr{H}$ is defined by

$$\mathscr{H} F(x) \equiv \int_J dy\, F(y)\delta(f(y) - x) \tag{A.1}$$

where δ is Dirac's δ function. From (A.1), we have

$$\int_J dx\, F\big(f^{(n)}(x)\big)G(x) = \int_J dx\, F(x)\mathscr{H}^n G(x) \qquad (n = 0, 1, 2, \ldots) \tag{A.2}$$

Integration of (A.1) leads to

$$\int_J dx\, \mathscr{H} F(x) = \int_J dx\, F(x) \tag{A.3}$$

Therefore, if φ_λ is an eigenfunction of $\mathscr{H}$ with an eigenvalue λ, we have

$$\int_J dx\, \varphi_\lambda(x) = 0 \tag{A.4}$$

for $\lambda \neq 1$. The eigenfunction with $\lambda = 1$ is the density function of the ergodic invariant measure for f. If f is ergodic, then $|\lambda| \leqslant 1$.[23]

APPENDIX B. FORMULAS FOR THE TIME-CORRELATION FUNCTIONS

We give formulas for the calculation of time-correlation functions in such cases as (I), (II), (III) stated in Section 2. These formulas, however, hold on the condition mentioned just below, and therefore can be applied to other maps which are piecewise-linear, continuous, or discontinuous.

Let f be a map on the interval J into itself, and $\mathscr{H}$ be the Frobenius–Perron operator of f. The condition for the formulas is as follows: There exist N subintervals $J_1, J_2, \ldots, J_N$ such that

i. J_l's are disjoint with each other except at most one point,

ii. the N-dimensional vector space spanned by $\theta_1(x), \theta_2(x), \ldots, \theta_N(x)$ is closed with respect to $\mathscr{H}$, where $\theta_l(x)$ is the indicator function of J_l,

iii. $\mathscr{H}$ has N different eigenvalues in this space.

Then the invariant density $\rho(x)$ in the interval J takes the form

$$\rho(x) = \sum_{l=1}^{N} d_l \theta_l(x) \tag{B.1}$$

as in (2.22), (2.25), and (2.28). The time-correlation function is written as (B.17) with (B.18) and (B.19) below. We first derive these formulas.

Let us define the column vectors

$$\boldsymbol{\theta}(x) \equiv \begin{bmatrix} \theta_1(x) \\ \theta_2(x) \\ \vdots \\ \theta_N(x) \end{bmatrix} \quad \text{and} \quad \mathscr{H}\boldsymbol{\theta}(x) \equiv \begin{bmatrix} \mathscr{H}\theta_1(x) \\ \mathscr{H}\theta_2(x) \\ \vdots \\ \mathscr{H}\theta_N(x) \end{bmatrix} \tag{B.2}$$

Then, by the condition (ii), we have

$$\mathscr{H}\boldsymbol{\theta}(x) = H_0 \boldsymbol{\theta}(x) \tag{B.3}$$

where $H_0 = (\xi_{ij})$ is an $N \times N$ matrix, the eigenvalues of which are denoted by $\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}$. By the condition (iii), there exists an $N \times N$ matrix U_0 such that

$$U_0^{-1} H_0 U_0 = D_0 = \operatorname{diag}(\lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_N^{(0)}) \tag{B.4}$$

where D_0 is a diagonal matrix. The elements u_{ij} and $\bar{u}_{ij}$ of the matrices $U_0 = (u_{ij})$ and $U_0^{-1} = (\bar{u}_{ij})$ are expressed in terms of ξ_{ij} and $\lambda_j^{(0)}$ by use of $H_0 U_0 = U_0 D_0$ and $U_0^{-1} H_0 = D_0 U_0^{-1}$, respectively. The invariant density $\rho(x)$ is a unique eigenfunction of $\mathscr{H}$ with the eigenvalue equal to unity. If we put $\lambda_1^{(0)} = 1$, we have $d_l = \bar{u}_{1l}$ $(l = 1, 2, \ldots, N)$.

Using (B.1), we have for the average of x^k

$$\langle x^k \rangle \equiv \int_J dx\, x^k \rho(x) = \left[U_0^{-1} \mathbf{J}^{(k)} \right]_1, \qquad (k = 0, 1, 2, \ldots) \tag{B.5}$$

where

$$\mathbf{J}^{(k)} \equiv \int_J dx\, x^k \boldsymbol{\theta}(x) = \begin{bmatrix} J_1^{(k)} \\ J_2^{(k)} \\ \vdots \\ J_N^{(k)} \end{bmatrix}$$
$$J_l^{(k)} = \int_J dx\, x^k \theta_l(x) = \int_{J_l} dx\, x^k \tag{B.6}$$

and $[U_0^{-1}\mathbf{J}^{(k)}]_j$ means the jth element of the vector $U_0^{-1}\mathbf{J}^{(k)}$. The eigenfunctions with the eigenvalues $\lambda_j^{(0)}$ are given by $\boldsymbol{\varphi}^{(0)} = U_0^{-1}\boldsymbol{\theta}$. Since $\lambda_j^{(0)} \neq 1$ for $j \neq 1$, we get from (A.4)

$$\int_J dx\, \varphi_j^{(0)}(x) = \left[U_0^{-1} \mathbf{J}^{(0)} \right]_j = \delta_{1,j} \tag{B.7}$$

Using (A.2) and (B.1), we have for $n = 0, 1, 2, \ldots$

$$\langle f^{(n)}(x) \cdot x \rangle \equiv \int_J dx\, f^{(n)}(x) x \rho(x) = \int_J dx\, x\left[U_0^{-1} \mathscr{H}^n \{ x\boldsymbol{\theta}(x) \} \right]_1 \tag{B.8}$$

If we define the $N \times N$ matrices H_1 and H_{10} by

$$\mathscr{H}\{ x\boldsymbol{\theta}(x) \} = H_1\{ x\boldsymbol{\theta}(x) \} + H_{10}\boldsymbol{\theta}(x) \tag{B.9}$$

then we have

$$\mathscr{H}\boldsymbol{\theta}^{(1)}(x) = H_1 \boldsymbol{\theta}^{(1)}(x) \tag{B.10}$$

where

$$\boldsymbol{\theta}^{(1)}(x) = x\boldsymbol{\theta}(x) - V\boldsymbol{\theta}(x) \tag{B.11}$$

V being the $N \times N$ matrix determined by

$$VH_0 - H_1 V = H_{10} \tag{B.12}$$

It follows from (B.10), (B.11), and (B.3) that $\mathscr{H}^n\{x\boldsymbol{\theta}(x)\} = VH_0^n\boldsymbol{\theta}(x) + H_1^n\boldsymbol{\theta}^{(1)}(x)$. Thus we have from (B.8)

$$\langle f^{(n)}(x) \cdot x \rangle = \left[U_0^{-1} V H_0^n \mathbf{J}^{(1)} + U_0^{-1} H_1^n (\mathbf{J}^{(2)} - V\mathbf{J}^{(1)}) \right]_1 \tag{B.13}$$

Since the eigenvalues of H_1 are not equal to unity, we get from (A.4)

$$\int_J dx\, \boldsymbol{\theta}^{(1)}(x) = \mathbf{J}^{(1)} - V\mathbf{J}^{(0)} = 0 \tag{B.14}$$

This leads to $U_0^{-1}\mathbf{J}^{(1)} = U_0^{-1}VU_0U_0^{-1}\mathbf{J}^{(0)}$, and therefore by use of (B.7)

$$\left[U_0^{-1}\mathbf{J}^{(1)}\right]_j = \left(U_0^{-1}VU_0\right)_{j1} \tag{B.15}$$

where $(U_0^{-1}VU_0)_{ij}$ means the ij-element of the matrix $U_0^{-1}VU_0$. Equation (B.15) and the fact that $\langle f^{(n)}(x)\rangle = \langle x\rangle$ yield

$$\langle x\rangle = \left[U_0^{-1}H_0^n\mathbf{J}^{(1)}\right]_1 = \left[U_0^{-1}\mathbf{J}^{(1)}\right]_1 = \left(U_0^{-1}VU_0\right)_{11} \tag{B.16}$$

From (B.13), (B.4), (B.15), and (B.16), the time correlation function is written as

$$C(n) \equiv \langle f^{(n)}(x)\cdot x\rangle - \langle x\rangle^2 = G_0(n) + G_1(n) \tag{B.17}$$

where

$$G_0(n) \equiv \left[U_0^{-1}VH_0^n\mathbf{J}^{(1)}\right]_1 - \langle x\rangle^2$$

$$= \sum_{j=2}^{N}\left(U_0^{-1}VU_0\right)_{1j}\left(\lambda_j^{(0)}\right)^n\left(U_0^{-1}VU_0\right)_{j1} \tag{B.18}$$

$$G_1(n) \equiv \left[U_0^{-1}H_1^n\left(\mathbf{J}^{(2)} - V\mathbf{J}^{(1)}\right)\right]_1 \tag{B.19}$$

We use (B.19) in Section 3, because the eigenvalues of the matrix H_1 are degenerate for the tent map at the band-splitting point $\bar{a}_m$.

If the condition that

iv. $\mathscr{H}$ has N different eigenvalues in the N-dimensional vector space spanned by $\theta_1^{(1)}(x), \theta_2^{(1)}(x), \ldots, \theta_N^{(1)}(x)$

is satisfied in addition to (i), (ii), and (iii), then the matrix H_1 is diagonalizable:

$$U_1^{-1}H_1U_1 = D_1 = \mathrm{diag}\left(\lambda_1^{(1)}, \lambda_2^{(1)}, \ldots, \lambda_N^{(1)}\right) \tag{B.20}$$

and we have from (B.19)

$$G_1(n) = \sum_{j=1}^{N}\left(U_0^{-1}U_1\right)_{1j}\left(\lambda_j^{(1)}\right)^n\left[U_1^{-1}\left(\mathbf{J}^{(2)} - V\mathbf{J}^{(1)}\right)\right]_j \tag{B.21}$$

The matrices $U_1 = (v_{ij})$ and $U_1^{-1} = (\bar{v}_{ij})$ are expressed in terms of $\lambda_j^{(1)}$ and $H_1 = (\eta_{ij})$. From (B.12) we also have

$$\left(U_1^{-1}VU_0\right)_{ij} = \left(U_1^{-1}H_{10}U_0\right)_{ij}/\left(\lambda_j^{(0)} - \lambda_i^{(1)}\right) \tag{B.22}$$

For the tent map at $a = b_{mK}$ and $a = c_{mK}$, we have $N = 2^m(K-1)$, and the N eigenvalues of H_1 are different from each other. Then (B.21) can be rewritten in a form which is more convenient for a systematic calculation. Thus the time-correlation function is given by

$$\frac{C(n)}{C(0)} = \sum_{j=2}^{N} A_j^{(0)}(\lambda_j^{(0)})^n + \sum_{j=1}^{N} A_j^{(1)}(\lambda_j^{(1)})^n \tag{B.23}$$

$$C(0) = \langle x^2\rangle - \langle x\rangle^2 = \langle x\rangle\left[a(a+1)^{-1} - \langle x\rangle\right] \tag{B.24}$$

$$A_j^{(0)} = (U_0^{-1}VU_0)_{1j}(U_0^{-1}VU_0)_{j1}/C(0) \tag{B.25}$$

$$A_j^{(1)} = (U_0^{-1}U_1)_{1j}\left[U_1^{-1}(\mathbf{J}^{(2)} - V\mathbf{J}^{(1)})\right]_j/C(0) \tag{B.26}$$

$$(U_0^{-1}VU_0)_{ij} = \frac{1}{2}\sum_{l=1}^{N}(U_0^{-1}U_1)_{il}\frac{\lambda_j^{(0)} - a\lambda_l^{(1)}}{\lambda_j^{(0)} - \lambda_l^{(1)}}(U_1^{-1}U_0)_{lj} \tag{B.27}$$

$$\left[U_1^{-1}(\mathbf{J}^{(2)} - V\mathbf{J}^{(1)})\right]_j = \sum_{l=1}^{N}(U_1^{-1}U_0)_{jl}\left[a\frac{\lambda_l^{(0)} - a\lambda_1^{(0)}}{\lambda_l^{(0)} - a^2\lambda_1^{(0)}} - \frac{1}{2}\frac{\lambda_l^{(0)} - a\lambda_j^{(1)}}{\lambda_l^{(0)} - \lambda_j^{(1)}}\right](U_0^{-1}VU_0)_{l1} \tag{B.28}$$

as shown in the following.

These formulas indicate that only the matrices $U_0^{-1}U_1$ and $U_1^{-1}U_0$ are needed for the calculation of $C(n)$, which are expressed in terms of the matrices H_0, H_1 and their eigenvalues $\lambda_j^{(0)}$, $\lambda_j^{(1)}$. The summations in (B.27) and (B.28) over the eigenvalues can be easily done by use of the formula

$$\sum_{l=1}^{N}\frac{1}{x - \lambda_l} = \frac{g'(x)}{g(x)} \tag{B.29}$$

where $g(x)$ is a polynomial of degree N and λ_l are the N roots of the equation $g(x) = 0$.

Equations (B.25) and (B.26) follow directly from (B.18) and (B.21). We now derive (B.27). In view of (4.1) and (4.10), we may write as

$$f_a J_l = \sum_{j=1}^{N}\sigma_{lj}J_j \tag{B.30}$$

where $\sigma_{lj} = 0$ or 1. Then we have for $H_0 = (\xi_{ij})$, $H_1 = (\eta_{ij})$ and $H_{10} = (\zeta_{ij})$

$$\xi_{ij} = \sigma_{ij}/a \tag{B.31}$$

$$\eta_{ij} = \begin{cases}\sigma_{ij}/a^2 & \text{if } J_i \subset [0, 1/2] \\ -\sigma_{ij}/a^2 & \text{if } J_i \subset [1/2, 1]\end{cases} \tag{B.32}$$

and

$$\zeta_{ij} = \begin{cases} 0 & \text{if } J_i \subset [0, 1/2] \\ \sigma_{ij}/a & \text{if } J_i \subset [1/2, 1] \end{cases} \tag{B.33}$$

Hence

$$H_{10} = (H_0 - aH_1)/2 \tag{B.34}$$

Thus we get from (B.22)

$$\left(U_1^{-1}VU_0\right)_{ij} = \frac{1}{2}\left(U_1^{-1}U_0\right)_{ij} \frac{\lambda_j^{(0)} - a\lambda_i^{(1)}}{\lambda_j^{(0)} - \lambda_i^{(1)}} \tag{B.35}$$

Since $U_0^{-1}VU_0 = (U_0^{-1}U_1)(U_1^{-1}VU_0)$, the expression (B.27) is obtained immediately.

In order to derive (B.28), we first notice that

$$\mathscr{H}\left\{x^2\boldsymbol{\theta}(x)\right\} = a^{-2}H_0\left\{x^2\boldsymbol{\theta}(x)\right\} - 2a^{-1}H_{10}\{x\boldsymbol{\theta}(x)\} + H_{10}\boldsymbol{\theta}(x) \tag{B.36}$$

Therefore, for the N-dimensional vector space spanned by

$$\boldsymbol{\theta}^{(2)}(x) = x^2\boldsymbol{\theta}(x) - V_1\boldsymbol{\theta}^{(1)}(x) - V_0\boldsymbol{\theta}(x) \tag{B.37}$$

we have

$$\mathscr{H}\boldsymbol{\theta}^{(2)}(x) = a^{-2}H_0\boldsymbol{\theta}^{(2)}(x) \tag{B.38}$$

where V_1 and V_0 are $N \times N$ matrices determined by

$$a^{-2}H_0V_1 - V_1H_1 = 2a^{-1}H_{10} \tag{B.39}$$

and

$$a^{-2}H_0V_0 - V_0H_0 = 2a^{-1}H_{10}V - H_{10} \tag{B.40}$$

respectively. The property (A.4) leads to

$$\int_J dx\, \boldsymbol{\theta}^{(2)}(x) = \mathbf{J}^{(2)} - V_0\mathbf{J}^{(0)} = 0 \tag{B.41}$$

From (B.40), (B.34), and (B.35) we have

$$\left(U_0^{-1}V_0U_0\right)_{ij} = a\frac{\lambda_i^{(0)} - a\lambda_j^{(0)}}{\lambda_i^{(0)} - a^2\lambda_j^{(0)}}\left(U_0^{-1}VU_0\right)_{ij} \tag{B.42}$$

Since $U_1^{-1}(\mathbf{J}^{(2)} - V\mathbf{J}^{(1)}) = (U_1^{-1}U_0)(U_0^{-1}V_0U_0)U_0^{-1}\mathbf{J}^{(0)} - (U_1^{-1}VU_0)$ $U_0^{-1}\mathbf{J}^{(1)}$, we obtain (B.28), where use has been made of (B.7), (B.15), (B.35), and (B.42).

We also have

$$\langle x^2\rangle = \left(U_0^{-1}V_0U_0\right)_{11} = a(a+1)^{-1}\langle x\rangle \tag{B.43}$$

Here (B.5), (B.41), (B.7) have been used for the first equality, and (B.42), (B.16) for the second. This leads to (B.24).

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